PORTFOLIO OPTIMIZATION MODEL WITH PROSPECT THEORY INVESTOR PREFERENCES

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The thesis presents a study of Kahneman-Tversky behavioral model of portfolio investment on the basis of the preferences of an investor. Optimization problem of the securities portfolio is formulated through the maximization of the prospect theory utility function. New formulation of the problem of portfolio selection is compared with the traditional approach of Markowitz problem.

Investors tend to create portfolios to maximize their utility within a Markowitz setting [1]. Thus, each investor wants a portfolio where the expected utility combined with portfolio risk level would maximize their satisfaction along with minimizing risk. Since each investor may have a different opinion on the level of risk they are prepared to accept, this leads to different opinions on the optimal combination of return and risk. Therefore we need to take into account that in determining an optimal portfolio we have to capture the investors attitude to risk.

Prospect theory is one of the Economic theories which was created for the purpose of risk estimation involving gains and losses. Within this framework, the individuals estimate the losses and gains utility function subjectively. For this purpose they use the return level of the non-risk governmental bonds as a starting point. Kahneman and Tversky [2] experimentally obtained the utility (value) function which was dependent on the initial value deviation. This value function is convex for the gains and non-convex for the losses. This means risk aversion is associated with the cases of the gains and risk inclination with the losses. It is worth mentioning that the aforementioned function has the steepest gradient for losses.

Let the random variables $r_j,\ j=1,\ldots,n$, denote the return of asset $j,\ j=1,\ldots,n$, and let x_j be the weight of asset *j* held in the portfolio. Then, the set of all the investment portfolios becomes

$$X=\left\{x=(x_1,\ldots,x_n)\in \mathrm{R}^n : \sum_{j=1}^{n} x_j = 1,\ x_j \ge 0,\ j=1,\ldots,n\right\}. \tag{1}$$

The utility function for the portfolio $x\in X$ can be defined as

$$U(x)=u(r(x)), \tag{2}$$

where $r(x)=\sum_{j=1}^{n} r_j x_j$ is the portfolio's return.

Let *p* to be a numerical boundary, so that

1. if a portfolio return value is equal to this number, then it means that the investor obtains a zero gain,
2. if a portfolio return value is higher, then the investor considers that he gained from the portfolio investment,
3. if it is lower, then he lost.

The value of *p* is assigned by the investor's preferences.

Taking into account the psychological effects of Prospect Theory, we assume that the game estimation function *u* is asymmetric with respect to a certain point *r* = *p*, i.e. *u(r)* is defined and increases along the real line, and its second derivative is less that zero on the interval $p<r<\infty$ and larger that zero on the interval $-\infty<r<p$, i.e. we will assume that $u\in U_1$, where U_1 is the set of all infinitely differentiable on $(-\infty,p)\cup(p,\infty)$ and is semi-differentiable at point *p* functions, such that

$$u^{(1)}>0, u^{(2)}<0 \text{ on } (p,\infty), u^{(2)}>0 \text{ on } (-\infty,p) \tag{3}$$

Note that in the Markowitz theory $u\in U_2$,where denote the set of smooth functions with

$$u^{(1)}>0, u^{(2)}<0, \tag{4}$$

i.e. the second derivative of the utility function is negative.

Let $\bar{r}_j$ be the mean of r_j, i.e. $\bar{r}_j = \mathrm{E}(r_j)$.We define $\bar{r}(x)$ to be the expected portfolio return level.

Denote $r_{\min} = \min\{r(x) : x \in X\}, r_{\max} = \max\{r(x) : x \in X\}$.

If we expand (2) in the Taylor series near the point $\bar{r}(x) \neq p$, we obtain

$$u(r(x)) = u(\bar{r}(x)) + \sum_{k=1}^{\infty} u^{(k)}(r(x)) \frac{(r(x) - \bar{r}(x))^k}{k!}.$$

The expected value of the function $u(r(x))$ is

$$\mathbb{E}(u(r(x))) = u(\bar{r}(x)) + \sum_{k=1}^{\infty} u^{(k)}(\bar{r}(x)) \frac{\mathrm{E}(r(x) - \bar{r}(x))^k}{k!}.$$

Applying the method of moments to the last equation we can get the structure of the optimal portfolio. The Markowitz model suggests that it is enough to use the first two moments.

The main goal of this study is to compare portfolio structures obtained using the two utility functions defined by (3) and (4). Rather than using behavioral setting as a simulation model (to understand investor behavior), we are using it as a decision model (to optimize portfolio with asymmetric investor preference). The problem was first examined in [3]. A clear connection between human behavioural biases and risk management in portfolio management has been found by Kahneman and Tversky. Traditional approach to portfolio management do not create opportunities for accounting all kind of risks and for choosing an optimal portfolio. As traditional optimization techniques cannot deal reliably with the extended problem, we suggest the use of a heuristic approach. It has been found that loss aversion has a substantial impact on what investors consider to be an efficient portfolio and that mean-variance analysis alone can be utterly misguiding.

In this presentation we provide results illustrating the effect of taking sentiment into account when constructing a portfolio. We analyze this impact with respect to a benchmark mean-variance model and provide both in-sample and out-of-sample statistics**.** The traditional approach [1] is to use increasing concave function as utility function and suggests that it is enough to use the first two moments (momentums).

Taking into account behavioural aspects of attitude of investors to losses [2] leads to the consideration of the problem

$$\mathbb{E}(u(r(x))) \to \max_{x \in X}, \tag{5}$$

where

$$u(r) = \begin{cases} (r - r_0)^{\alpha}, r \geq r_0, \\ \lambda (r - r_0)^{\beta}, r < r_0, \end{cases} \tag{6}$$

r_0 is specified rate of return, α, β, λ is positive constants, which characterize the attitude of the investor to losses.

As problem (5), (6) is not convex and standard approaches of non-linear optimization can not guarantee its solution. In this connection we use some heuristic optimization algorithms [4], [5].

The empirical study is based on the stocks included in the FTSE100. Using adjusted daily prices downloaded fromfinance.yahoo.com for 1 January 2009 to 24 December 2011, we will show that taking into account investor's loss aversion has significant impact on the structure of the optimal portfolio which has significant differences with the structure of optimal portfolio of the resulting based on Markowitz's approach. We assume that our results help to understand seemingly irrational behavior and underline the inadequacy of the mean-volatility framework for many real life investment problems.

References

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