Quantitative Finance

ISSN: 1469-7688 (Print) 1469-7696 (Online) Journal homepage: http://www.tandfonline.com/loi/rquf20

Prospect theory–based portfolio optimization: an empirical study and analysis using intelligent algorithms

N. Grishina, C. A. Lucas & P. Date

Quantitative Finance, 2016
http://dx.doi.org/10.1080/14697688.2016.1149611

Prospect theory–based portfolio optimization: an empirical study and analysis using intelligent algorithms

N. GRISHINA*, C. A. LUCAS and P. DATE

College of Engineering, Design and Physical Sciences, Department of Mathematics, Brunel University, Kingston Lane, Uxbridge, Middlesex UB8 3PH, UK

(*Received 4 March 2014; accepted 28 January 2016; published online 18 July 2016*)

The behaviourally based portfolio selection problem with investor's loss aversion and risk aversion biases in portfolio choice under uncertainty is studied. The main results of this work are: developed heuristic approaches for the prospect theory model proposed by Kahneman and Tversky in 1979 as well as an empirical comparative analysis of this model and the index tracking model. The crucial assumption is that behavioural features of the prospect theory model provide better downside protection than traditional approaches to the portfolio selection problem. In this research the large-scale computational results for the prospect theory model have been obtained for real financial market data with up to 225 assets. Previously, as far as we are aware, only small laboratory tests (2–3 artificial assets) have been presented in the literature. In order to investigate empirically the performance of the behaviourally based model, a differential evolution algorithm and a genetic algorithm which are capable of dealing with a large universe of assets have been developed. Specific breeding and mutation, as well as normalization, have been implemented in the algorithms. A tabulated comparative analysis of the algorithms' parameter choice is presented. The prospect theory model with the reference point being the index is compared to the index tracking model. A cardinality constraint has been implemented to the basic index tracking and the prospect theory models. The portfolio diversification benefit has been found. The aggressive behaviour in terms of returns of the prospect theory model with the reference point being the index leads to better performance of this model in a bullish market. However, it performed worse in a bearish market than the index tracking model. A tabulated comparative analysis of the performance of the two studied models is provided in this paper for in-sample and out-of-sample tests. The performance of the studied models has been tested out-of-sample in different conditions using simulation of the distribution of a growing market and simulation of the t-distribution with fat tails which characterises the dynamics of a decreasing or crisis market.

Keywords: Portfolio optimization; Behavioural finance; Prospect theory; Index tracking; Risk modelling

JEL Classification: C61, C63

1. Introduction

The portfolio optimization problem addresses the question of how to determine an amount (proportion, weight) of money to invest in each type of asset within the portfolio in order to receive the highest possible return (or utility) subject to an appropriate level of risk by the end of the investment period.

Modern portfolio theory (MPT) began with a paper (Markowitz 1952) and a book (Markowitz 1959) written by the Nobel laureate Harry Markowitz. Many researchers consider the emergence of this theory as the birth of modern financial mathematics (Rubinstein 2002). The cornerstones of Markowitz's theory are the concepts of return, risk and diversification. It is widely accepted (Rubinstein 2002) that an investment portfolio is a collection of income-producing assets that have been acquired to meet a financial goal. However, an investment portfolio as a concept did not exist before the late 1950s.

Remarkably, there is a long history behind the expected utility theory (EUT) that started in 1738 when Daniel Bernoulli investigated the St. Petersburg paradox. He was the first scientist who separated the definitions of 'price' and 'utility' in terms of determining the item's value. Price is an assessment of an item and depends only on the item itself and its

*Corresponding author. Email: grishinaninapavlovna@gmail.com

characteristics, i.e. price is the objective value. In contrast, utility is subjective and 'is dependent on the particular circumstances of the person making the estimate' (Bernoulli 1954). EUT follows the assumptions of the neoclassical theory of individual choice in cases when risk appears. It was formally developed by John von Neumann and Oscar Morgenstern in their book 'Theory of Games and Economic Behavior' (1944) (Neumann and Morgenstern 1944).

The theory's main concern is the representation of individual attitudes towards risk (Karni 2014). Since the 1950s, several papers appeared showing that the empirical evidence on individuals' patterns of choice under risk are inconsistent with the expected utility theory (Pirvu and Schulze 2012). It is also shown (Rieger and Wang 2008) that the players' behaviour systematically violates the independence axiom which states that the preference relations between 2 outcomes (lotteries) will state the same even if each of these outcomes (lotteries) will mixed with a third outcome (lottery) with the same probability p. Violation of this axiom is also known as 'common consequence' problem. At the same time the EUT is unable to explain many paradoxes that take place in economic practice (for example, the Allais Paradox (Allais 1953)).

The number of EUT's drawbacks led to the appearance of the behavioural portfolio theory (BPT)—a new fundamental framework which was designed to compensate for the misguidings of the EUT. To date it is the best model for explaining the behaviour of the players and investors in an experiment in decision-making under risk. In contrast to EUT, BPT fills in some gaps in explaining controversial economic phenomena, such as the Ellsberg Paradox (Ellsberg 1961).

The recent financial crisis has shown the shortcomings of the individual market instruments and the low level of validity in investment decisions. This can be explained by the dismissive investors' attitude in assessing the real risks, they usually just follow their own intuition. In investment practice, the situation of unaccounted risks is fairly common, hence, the investors need to have a reliable mathematical tool for justification of investment decisions. In this paper, we consider BPT as a tool which takes into account behavioural errors.

BPT was developed by Shefrin and Statman (2000). The main idea of the theory is the maximization of the value of the investor's portfolio in which several goals are met and these goals are considered with different levels of risk aversion. BPT is based on two main theories: security-potential/aspiration theory (SP/A) and prospect theory (PT). SP/A theory, established by Lola Lopez 1987, is a general choice (not only financial) risk framework and not specified for the portfolio selection problem. It uses two independent criteria of choice—overall utility and aspiration level. In our research, we focus on the PT (Kahneman and Tversky 1979) devoted to human behaviour in financial decision-making under uncertainty.

PT adopts the main idea from the EUT and adds in the vital psychological components, which take into account human behaviour in the decision-making process. It also fixes different types of inaccuracies that took place in previously developed behaviour-based theories, e.g the independence axiom and inconsistency with a uniform attitude towards risk, see Shefrin and Statman (2000).

Loss aversion is a corner stone for prospect theory, especially for portfolio performance evaluation (Zakamouline and Koekebakker 2009) and market price of risk (Levy 2010). Since prospect theory was proposed many researchers studied the loss aversion effect in asset pricing (Barberis and Huang 2001, Lia and Yang 2013, Easley and Yang 2015), price volatility (Yang and Wu 2011) and insurance (Wang and Huang 2012) very successfully. As far as we aware, despite many papers devoted to PT, few researchers have investigated the portfolio selection problem with prospect theory and index tracking in terms of diversification and return performance. We also have compared the prospect theory model with index tracking to the original index tracking model.

The goal of this paper is to identify potential benefits of behaviourally based prospect theory model depending on different market situations in comparison with traditionally accepted portfolio optimization model such as index tracking (IT) model. In this paper, we apply the PT model to several empirical and experimental data-sets in order to find an optimal solution to the portfolio selection problem with index tracking settings. In order to do so we develop appropriate solution approaches to prospect theory namely genetic algorithm and differential evolution algorithm which take into account mathematically complexity of the researched problem. We also test the results out-of-sample and compare the performance of the PT model with the results obtained by the index tracking model. We investigate these models performance also with a cardinality constraint. The main contribution of our work is large-scale computational results using metaheuristics obtained for the prospect theory portfolio selection problem for data from various financial markets, with the asset universe of each ranging from 31 assets to 225 assets.

2. Literature review

Prospect theory is a behavioural economic theory that describes decisions between alternatives that involve risk, where the probabilities of outcomes are known. It was developed as a descriptive model of decision-making under uncertainty by two psychologists, Daniel Kahneman and Amos Tversky, and published in the Econometrica in 1979 (Kahneman and Tversky 1979). The authors relied on a series of small experiments to identify the manner in which people make choice in the face of risk. The theory says that people make decisions based on the potential value of losses and gains rather than the final outcome, and that people evaluate these losses and gains using heuristics. Although the original formulation of prospect theory was only defined for lotteries with two non-zero outcomes, it can be generalized to n outcomes. Generalizations have been used by various authors Schneider and Lopes (1986), Camerer and Ho (1994), Fennema and Wakker (1997), Vlcek (2006).

The original PT choice process consists of two phases. During the first phase, which is called editing, an agent defines their own (subjective) meaning of a gain and a loss by setting a reference point r_0 for the portfolio return, which represents zero gain (or zero loss) for this particular person. During the second stage, which is called the evaluating phase, our investor calculates the values of the prospect theory utility based on the potential outcomes and their respective probabilities, and chooses the maximal one.

To understand the features of prospect theory let us analyse two approaches to the portfolio selection problem which are traditional (MPT) and behavioural (behavioural portfolio theory). We focus more on the assumptions underlying these theories which govern the investor's choice.

MPT uses several basic assumptions namely 'rational investor', normal distribution of asset returns and neglection of transaction costs (Markowitz 1959). At the same time it was shown that in real-life market conditions, these assumptions are not valid (Mandelbrot 1963, Fama 1968, Evans and Archer 1968, Fisher and Lorie 1970, Jacob 1974, Szego 1980, Patel and Subrahmanyam 1982, Sengupta and Sfeir 1985, Peng *et al*. 2008, Das *et al*. 2010).

The mean variance model which exists in the MPT framework is both sufficiently general and static for a significant range of practical situations and at the same time it is simple enough for theoretical analysis and numerical solution. This benefit provides widely use of the mean variance model in practice all over the world. However, the portfolio selection problem becomes even more complicated in modern economic conditions which demand more flexible and multi-factor models and tools to satisfy the investor's preferences, while MPT's assumptions lead to some serious limitations.

The question about the difference and ratio between the portfolio allocation according to mean variance optimization and prospect theory utility function optimization is very challenging in the literature. Many scientists attempt to conceptualize the benefits and drawbacks of each approach depending on specific market situations, data and assumptions. There are several reasons why it is not easy to compare MPT and prospect theory approaches.

The first obstacle can be called computational difficulties. Due to the fact that the PT model is very complex from a computational (solution approach) point of view, only simple cases for the analysis are available in the literature. For example, many researchers used only 2–3 assets to get the portfolio allocation based on prospect theory (Kahneman and Tversky 1979). However, it is not enough for rigorous comparative analysis.

The first effort to compare two models was made by Levy and Levy Levy and Levy (2004). The idea was to select the portfolio with the highest prospect theory utility among the other portfolios in the mean variance efficient frontier. Following this route Pirvu and Schulze in 2012 present results confirming that an analytical solution is mostly equivalent to maximising the PT objective function along the mean variance efficient frontier (Pirvu and Schulze 2012).

The next step in development was connected with application to general return distributions. The attempts started with an application to a market with two assets available: one of them is the risk-free and the other is a risky asset. As a computational approach for the problem, the piecewise-power value function is considered. Originally this method was suggested by Tversky and Kahneman Tversky and Kahneman (1992). Gomes in 2005 applied this idea to prospect theory (Gomes 2005). For more information about the piecewise-quadratic approach to PT see Hens and Bachmann (2008), Zakamouline and Koekebakker (2009). Many researches propose a heuristic approach as an effective tool for dealing with non-convex problems (Maringer 2008).

Metaheuristic approach is very popular method for solving the portfolio selection problem, in a constrained formulation which is NP-hard and difficult to be solved by standard optimization methods (Gaspero *et al*. 2011, di Tollo 2015). Adebiyi Ayodele and Ayo Charles used metaheuristics method of generalized differential evolution three in order to solve extended Markowitz mean variance portfolio selection model consists of four constraints: bounds on holdings, cardinality, minimum transaction lots and expert opinion (Ayodele and Charles 2015). Other researches devoted to using metaheuristics for solving constrained portfolio selection problem see in the following sources Lin and Wang (2002), Lin and Liu (2008), Dueck and Winker (1992).

In this paper, we propose two heuristic approaches to the prospect theory portfolio selection problem: the differential evolution algorithm and the genetic algorithm. A recent addition to the class of evolutionary heuristics is proposed by Storn and Price 1997, Price *et al*. (2005) which is based on the evolutionary principle using differential weight (F) as a mutation factor. This solution approach has been used by Maringer (2008) who studied PT investor's risk aversion and loss aversion using higher order moments such as skewness and kurtosis. To best of our knowledge he was the first researcher who adopted this algorithm to the behaviourally based optimization problem.

A genetic algorithm is a searching mechanism which is based on evolutionary principles of natural selection and genetics. The theoretical background of genetic algorithms was developed by Holland. It works with populations of solutions and uses the principles of survival of the fittest. In genetic algorithms the variables of the solution are coded into chromosomes (Holland 1975). To make a natural selection and get good solutions, chromosomes are evaluated by a fitness criterion. In the considered optimization problems the measure of fitness is usually connected with the objective function (Mitchell 1996, Beasley 2002, Aarts *et al*. 2003). As far as we are aware, the genetic algorithm has not been applied to the prospect theory problem.

Later several approaches to get computational results for the prospect theory utility function optimization were developed (Levy and Levy 2004, Pirvu and Schulze 2012). Then the question about which data should be used arose. Most of the researches are based on normally distributed testing data (Levy and Levy 2004). At the same time it is well known that many asset allocation problems involve non-normally distributed returns since commodities typically have fat tails and are skewed (Mandelbrot 1963, Fama 1968). Therefore, in our research we obtain and test the optimal portfolios on several sets of data including data with a t-distribution and bearish market data.

According to the problem formulation and theoretical basis the mean variance model manages the risk of the portfolio taking into account the covariance matrix and standard deviation of assets. Modern portfolio theory and the work of Harry Markowitz on diversification and risk of a portfolio established the capital asset pricing model (CAPM) which distinguishes two types of portfolio risk: systematic and unsystematic. Systematic risk is considered as a market risk, i.e. it is undiversifiable and common for all assets in the market, while unsystematic risk is associated with each security. In terms of CAPM the optimal portfolio which aims to achieve the lowest risk together with any possible return is the market portfolio which, in fact, could be a market index. Following the assumption of CAPM the index tracking problem for portfolio selection is a replication of the 'ideal' market portfolio in order to reduce unsystematic risk.

Index tracking, known as a form of passive fund management, aims to produce optimal portfolios which replicate the index dynamics providing a balance between risk and return. However, the index tracking model normally includes almost all available assets in the market that leads to large transaction costs and a portfolio which is very difficult to manage because of its diversity (Beasley *et al.* 2003). Thus, the cardinality constrained index tracking model is also considered in this paper. We explore this model in comparison with behaviourally based model (the prospect theory model) in terms of diversification and tracking error (TE) issues.

Generally speaking, MPT uses a model that attempts to describe how capital markets operate, not a recipe for designing investment portfolios. Curtis in his paper 'MPT and Behavioral Finance' assumed that MPT 'is very useful, but it is descriptive, not prescriptive, and relies on assumptions that may not always be valid' (Curtis 2004). It is a very limited theory in terms of application to the real economic world conditions (Shefrin and Statman 2000, Shefrin 2001). In contrast, BPT gives us flexibility and range of tools such as natural investor preferences (risk aversion, loss aversion, etc.) which provide an opportunity to investigate and to adjust risk component in portfolio selection more deeply and precisely.

3. Problem formulation

In this section, we formulate the index tracking and the prospect theory with index tracking models with cardinality constraint.

First we set out some general notation that we use for all of our models. In this section and in the rest of the paper, we will use the following notation:

N—number of assets,
S—number of scenarios (time periods),
K—cardinality limit (desirable number of assets in the portfolio),
p_s—probability of scenario s, $\sum_s p_s = 1$,
$\bar{r}_i$—mean return of asset i,
r_{is}—return of asset i in scenario s, $i = 1, \ldots, N$, $s = 1, \ldots, S$,
r_0—reference point,
$\omega_i \geq 0$—weight of asset i in the portfolio,
$x = (\omega_1, \ldots, \omega_N)$—a portfolio and $\sum_{i=1}^{N} \omega_i = 1$,
$X = \{x = (\omega_1, \ldots, \omega_N) \in \mathbb{R}_+^N\}$—set of all portfolios,
$r_s(x)$—return of portfolio x in scenario s,
d—desirable level of return.

It should be noted that one can transfer these models with a cardinality constraint into the basic models if we put $K = N$. For the sake of simplicity we can use a unified formulation for both, basic and cardinality constrained models.

3.1. *Index tracking model*

In our research we use a simple index tracking model in the form of full replication as we are minimizing the TE in order to reduce the difference between the index return and the portfolio return.

Let at time s

rm_s – index return,
$o_s = \max(r_s(x) - rm_s, 0)$ – portfolio return amount over the index return,
$u_s = \max(rm_s - r_s(x), 0)$ – portfolio return amount under the index return.

TE for a given time period is equal to $|r_s(x) - rm_s|$. Clearly, at time s at least one of o_s or u_s is equal to 0, i.e. we can define a new quantity

$$\mathrm{TE}_s = o_s + u_s = \begin{cases} o_s, \text{if} o_s \geq 0, \\ u_s, \text{otherwise}. \end{cases} \quad (1)$$

Let us define the TE in the simplest possible way: as the difference between the index and portfolio returns over all time periods $s = 1, \ldots, S$:

$$\mathrm{TE} = \sum_{s=1}^{S} \mathrm{TE}_s. \quad (2)$$

Here, we would like to mention that TE can be defined in different ways, for example, in Roll (1993) the TE is defined as the root mean square of the difference between index and portfolio returns.

As was mentioned previously we can use the formulation of the cardinality constrained model for the basic model as well when we put $K = N$. Then the index tracking problem with cardinality constraint can be formulated as (Reilly and Brown 2005):

$$\text{minimize} \mathrm{IT}_{\mathrm{cc}}(x) = \text{minimize} \mathrm{TE}(x) = \sum_{s=1}^{S} (o_s + u_s), \quad (3)$$

subject to the constraints

$$\sum_{i=1}^{N} \omega_i r_{is} = rm_s + o_s - u_s, s = 1, \ldots, S \quad (4)$$

$$\sum_{i=1}^{N} \omega_i = 1, \quad (5)$$

$$l_i \varphi_i \leq \omega_i \leq u_i \varphi_i, \ i = 1, \ldots, N, \quad (6)$$

$$\sum_{i=1}^{N} \varphi_i \leq K, \quad (7)$$

$$\varphi_i \in \{0, 1\}, \ i = 1, \ldots, N, \quad (8)$$

$$\omega_i \geq 0, i = 1, \ldots, N, \quad (9)$$

$$o_s, u_s \geq 0, s = 1, \ldots, S. \quad (10)$$

Equation (4) checks the difference between returns of the optimal portfolio and the index for each time period. Constraint (5) imposes that the investment weights sum to one (budget constraint). Inequality (6) describes a buy-in threshold and restricts asset investment. It is easy to see that if an asset i is not held, i.e. $\varphi_i = 0$, then the corresponding weight $\omega_i = 0$. If an asset i is held, i.e. $\varphi_i = 1$, then (6) ensures that the value of ω_i lies between the appropriate lower and upper limits, l_i and u_i, respectively (Woodside-Oriakhi *et al.* 2011). Inequality (7) ensures that the number of assets in the optimal portfolio is

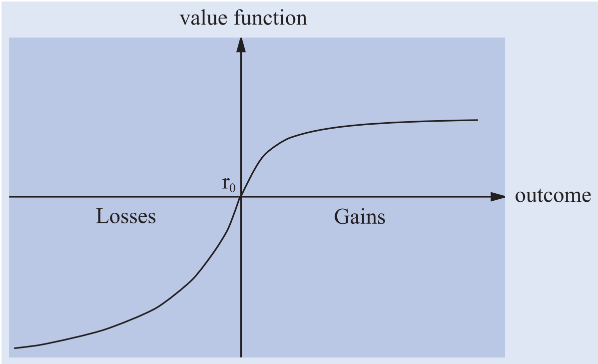

Figure 1. Prospect theory value function $v(r)$ with $\alpha = \beta = 0.88$ and $\lambda = 2.25$.

at most K. The binary definition (8) reflects the inclusion (or exclusion) of an asset in the portfolio.

3.2. Prospect theory model for index tracking

Consider the game:

$$(r_1, p_1), (r_2, p_2), \dots, (r_0, p_0), \dots, \\ \times (r_{S-1}, p_{S-1}), (r_S, p_S), \tag{11}$$

where (r_s, p_s), $s = 1, 2, \dots, 0, \dots, S-1, S$, means that the gambler wins r_s with probability p_s, of course, the sum of all probabilities is equal to 1, i.e. $\sum_{s=1}^{S} p_s = 1$; r_0 denotes some numerical boundary called the reference point (constant) which depends on the investor's preference. Let r_s define the outcomes of the game (11) such that:

- if $s = 0$, i.e. $r_s = r_0$, then the investor's gain is 0,
- if $s > 0$, then $r_s > r_0$, hence the investor won from this investment,
- if $s < 0$, then $r_s < r_0$, hence the investor lost.

According to the prospect theory one needs to make additional mental adjustments in the original probability and outcome value functions p and r, which is equivalent to replacing a standard utility function by the prospect theory utility function. In order to do so we transform the original p and r into the prospect theory probability weight function $\pi(p)$ and value function $v(r)$.

The prospect theory probability weighting function $\pi(p)$ measures, according to Kahneman and Tversky (1979), 'the impact of events on the desirability of prospects, and not merely the perceived likelihood of these events', i.e. expresses the weights of the decisions to the probabilities. Let us mention that $\pi(p)$ is an increasing function, $\pi(0) = 0$, $\pi(1) = 1$, and for very small values of probability p we have $\pi(p) \geq p$. The probability weighting function based on the observation that most people tend to overweigh small probabilities and underweigh large probabilities.

The prospect theory value function $v(r)$ describes the (behavioural) value of the gain/loss outcome. Kahneman and Tversky experimentally obtained the value function which was dependent on the initial value deviation. This function is usually asymmetric with respect to a given reference point r_0 (which reflects different investor's attitude to gains and losses), it is concave upward for gains and convex downward for losses. Moreover, generally the value function $v(r)$ grows steeper for losses than for gains, i.e. for $s > 0$ we have $v(r_s) \leq -v(r_{-s})$.

The explicit formula for the prospect theory value function $v(r)$, given in Tversky and Kahneman (1992), is:

$$v(r) = \begin{cases} (r - r_0)^{\alpha}, \text{if} \quad r \geq r_0, \\ -\lambda(r_0 - r)^{\beta}, \text{if} \quad r < r_0, \end{cases} \tag{12}$$

where $\alpha = \beta = 0.88$ are risk aversion coefficients with respect to gains and losses accordingly, $\lambda = 2.25$ is the loss aversion coefficient which underlines differences in the investor's perception of gains and losses. We note that the value function (12) is non-linear with respect to return r and, hence, the portfolio variable x. Figure 1 contains the graphs for the value function $v(r)$.

The prospect theory utility function can be written in terms of π and v as:

$$\mathrm{PT_U} = \sum_{s=1}^{S} \pi(p_s) v(r_s) = \sum_{s=1}^{S} p_s v \left(\sum_{i=1}^{N} r_{si} \omega_i \right). \tag{13}$$

Clearly, the formula (12) consists of two parts. The part in the gain domain (i.e. when $r \geq r_0$) is concave and the part in the loss domain (i.e. when $r \leq r_0$) is convex, capturing the risk-averse tendency for gains and risk-seeking tendency for losses as seen by many decision-makers (Rieger and Wang 2008). Let as mention, that for the sake of simplicity in our study we use $\pi(p) = p$. Clearly, the prospect theory utility function (13) is a non-linear function.

The prospect theory model aims to find the best (optimal) portfolio which maximizes the prospect theory utility function where decision variables are weights of available assets ω subject to constraints on a desirable level of return (in the case of basic prospect theory problem formulation), budget and short sales. This is a non-linear and non-convex optimization model as the objective function is non-linear and non-convex. In order to solve this problem we use heuristics which are an inexact solution approach.

According to the prospect theory portfolio selection problem looks as follows (basic prospect theory model):

$$\text{maximize} \mathrm{PT}(x) = \sum_{s=1}^{S} p_s v \left(\sum_{i=1}^{N} r_{si} \omega_i \right), \tag{14}$$

subject to the constraints

$$\bar{r}(x) = \sum_{i=1}^{N} \bar{r_i} \omega_i \geq d, \tag{15}$$

$$\sum_{i=1}^{N} \omega_i = 1, \tag{16}$$

$$\omega_i \geq 0, i = 1, \dots, N. \tag{17}$$

Studying the prospect theory problem we found that the principle of the model is very similar to that of the index tracking portfolio optimization problem. The main common feature is that behaviourally based models use a reference point as the limit for desired level of returns in each time period similar to an index tracking model which uses the index as a reference point. Thus it is easy to implement the idea of the index tracking problem into prospect theory by changing the value of the reference point. In this case, we let r_0 be a vector of the index value for each time period of the data-set not a scalar as it is in the original version of the prospect theory. We also remove the limit on the desirable level of returns similar to the index tracking problem which focuses on the index value as a level of return for each time period. We call this model prospect theory with index tracking (PT with IT).

We also implemented a cardinality constraint in this model to address the issue of too diversified a portfolio. It is very interesting to compare not only the IT and PT with index tracking problems but these models with the limit on the number of the assets in the portfolio. We formulate the prospect theory model with index tracking and with a cardinality constraint as:

$$\text{maximize } \mathrm{PT}+\mathrm{IT}_{\mathrm{cc}}(x)=\sum_{s=1}^{S} p_s v\left(\sum_{i=1}^{N} r_{si}\omega_i, rm_s\right), \quad (18)$$

subject to the constraints

$$\sum_{i=1}^{N}\omega_i = 1, \quad (19)$$

$$l_i\varphi_i \le \omega_i \le u_i\varphi_i,\ i=1,\dots,N, \quad (20)$$

$$\sum_{i=1}^{N}\varphi_i \le K, \quad (21)$$

$$\varphi_i \in \{0,1\},\ i=1,\dots,N, \quad (22)$$

where

$$v\left(r(x), rm_s\right)=\begin{cases}(r(x)-rm_s)^{\alpha}, \text{if} \quad r(x)\ge rm_s,\\ -\lambda(rm_s-r(x))^{\beta}, \text{if} \quad r(x)<rm_s.\end{cases} \quad (23)$$

As one can see in equation (23) the value function for the prosect theory model with index tracking is defined as a dynamic not constant due to the fact that instead of a constant reference point r_0 here we use a dynamic index rm which takes different values in each scenario (time period).

4. Solution approach for the prospect theory model

In the previous chapter, we considered two basic models: index tracking and prospect theory with index tracking models. The index tracking problem is mixed-integer linear problem and can be solved easily with a built-in solver. For the IT (also with cardinality constraint) in our empirical study we use the standard solver *CPLEX* (AMPL) which is developed to deal with integer, mixed-integer, linear programming and quadratic problems, including problems with quadratic constraints possibly involving integer variables. In contrast, the prospect theory model is non-convex. Hence, the solution approach becomes more challenging.

It is important to note that problem (18)–(22) is non-convex and function (18) is non-differentiable. In addition, we consider the cardinality constrained PT with IT model which potentially makes the problem more complex for solving. This complexity of the researched problem does not allow the use exact methods due to the increasing CPU time and restrictions of size of reasonable data-sets. As long as it is very difficult to find an optimal solution for this type of problem many researchers and traders use heuristics that are inexact methods to solve this sort of portfolio optimization problems.

In our research, we use two heuristic solution approaches for the cardinality constrained portfolio optimization problem with behavioural component. The first is based on the differential evolution algorithm proposed by Storn and Price (1997), Price *et al*. 2005) and adopted to the prospect theory model by Maringer (2008). In the development of paper (Chang *et al*. 2009), we suggest the genetic algorithm as the second approach to the researched problem which is based on meta-heuristic approach (Holland 1975) in order to find the 'optimal' solution for the prospect theory with index tracking cardinality constrained portfolio optimization problem.

It should be noted that we also tried to use tabu search and pattern search methods (in Matlab) and have to refuse the results due to the fact that found optimal portfolios by these algorithms consists only of 1 or 2 assets among 31 available. These portfolios do not appear to be interesting for real-market conditions and strategies.

For the sake of simplicity in our calculations we define the prospect theory weighting function as $\pi(p) = p$ and use the original value function $v(r)$ as proposed in Tversky and Kahneman (1992) using dynamic reference point rm as defined in (23).

4.1. *Differential evolution algorithm*

Let N be the number of all available assets. We need to find an optimal value of a uniformly distributed variable $x=(\omega_1,\omega_2,\dots,\omega_N)\in D\subseteq\mathbb{R}^N$, where D is a set of feasible objective function values, i.e. we are looking for the value of $x\in D$, which provides a solution for the problem (18). In order to find this optimal value of x we need to maximize the value of $\mathrm{PT}+\mathrm{IT}_{\mathrm{cc}}(x)$ (which is equivalent to $\mathrm{PT}+\mathrm{IT}(x)$ if $K=N$) using the following steps.

1. Initialization. We define the set

$$D_K = \{v\in D, \text{such that exactly} K \text{components of vector} v \text{are positive}\}.$$

Let $P\in\mathbb{N}$. We generate an initial population $x_i=(\omega_{i1},\dots,\omega_{iN}), \forall i=1,\dots,P^2 x_i\in D_K$.

2. Mutation and crossover. At each generation $g=1,\dots,G$ let take x_i and choose vectors x_a, x_b, x_c randomly from the population's vectors x_l, $l=1,\dots,P^2$, such that they do not coincide with x_i and each other. Also pick a random number $R\in\{1,\dots,N\}$. We construct the components of a new vector $\tilde{x}_i\in D$ as follows. With probability CR and if $R=j$, $j=1,\dots,N$, for the jth component, we assume $\tilde{x}_{ij}=v_{aj}+(F+z_1)(x_{bj}-x_{cj}+z_2)$ and $\tilde{x}_{ij}=x_{ij}$ otherwise. Here, parameters $F\in[0,2]$ and $\mathrm{CR}\in[0,1]$ are called the differential weight and the crossover probability, respectively, and should be chosen by the user; quantities z_1 and z_2 are either zero with a low probability (e.g. 0.0001 and 0.0002, respectively), or are normally distributed random variables with a mean of zero and a small standard deviation (for example 0.02). The parameters z_1 and z_2 are optional for the differential evolution algorithm. They are used to add up some 'noise' to the calculation of the resulting vector and avoid getting into local extrema.

3. Selection. Using equation (18) we calculate the values $\mathrm{PT}+\mathrm{IT}_{\mathrm{cc}}(x_i)$ and $\mathrm{PT}+\mathrm{IT}_{\mathrm{cc}}(\tilde{x}_i)$ and choose the maximum called $\max(x_i)$ to proceed to the new population which is used in the next generation until the stopping criteria (e.g. number of generations, precision, etc.) is met.

4. Final Assessment. In the last generation $g = G$ find the vector which $y_i = \{x_i \mid \max\{\mathrm{PT} + \mathrm{IT}_{cc}(x_1), \ldots, \mathrm{PT} + \mathrm{IT}_{cc}(x_{P^2})\}$. The vector y_i then is our best solution (Homchenko *et al.* 2013).

Pseudo-code of the differential evolution algorithm for prospect theory utility function maximization is given below.

```
Generate initial population x_i ∈ D_K, i = 1, ..., P^2,
cycle of G generations
    for each x_i in population P
    choose 3 random vectors x_a ≠ x_b ≠ x_c ≠ x_i
        for each component j of x_i do
            with probability π_1 : z_1 ← N(0, σ_1),
            else z_1 = 0
            with probability π_2 : z_2 ← N(0, σ_2),
            else z_2 = 0
            pick u_j ~ U(0, 1)
            if u_j < CR or j = R
                    then x̃_ij = x_aj + (F + z_1)
                    (x_bj − x_cj + z_2)
                    else x̃_ij = x_ij
        if PT + IT(x̃_i) > PT + IT(x_i)
            then ỹ_i = x̃_i
            else ỹ_i = x_i
    In g = G find y_i = {ỹ_i | max{PT+IT_cc(ỹ_1), ..., PT+
IT_cc(ỹ_P2)}}.
```

4.2. Genetic algorithm

To maximize the objective function or utility function $\mathrm{PT} + \mathrm{IT}_{cc}(x)$ given in formula (18) using a genetic algorithm we need to make the following steps.

1. Initialization. We define the set

$$D_K = \{x \in D, \text{ such that exactly } K \text{ components of vector } x \text{ are positive}\}.$$

Let $P \in \mathbb{N}$. We generate an initial population $x_i = (\omega_{i1}, \ldots, \omega_{iN}), \forall i = 1, \ldots, P^2 x_i \in D_K$.

2. Selection. At each generation $g = 1, \ldots, G$ we calculate values $\mathrm{PT}+\mathrm{IT}_{cc}(x_1), \ldots, \mathrm{PT}+\mathrm{IT}_{cc}(x_{P^2})$ and put them in decreasing order, i.e. we obtain a decreasing sequence

$$\left(\mathrm{PT} + \mathrm{IT}_{cc}(x_{m_1}) \geq \ldots \geq \mathrm{PT} + \mathrm{IT}_{cc}(x_{m_{P^2}})\right),$$

where set $x_{m_1}, \ldots, x_{m_{P^2}}$ is a permutation of the initial set $x_1, \ldots, x_{P^2}$. We fix the maximum value of the objective function $\max\mathrm{PT}+\mathrm{IT}_{cc}(x_i)$. Only the first $2P$ elements move to the new population without changes, i.e. $x_{m_1}, \ldots, x_{m_{2P}}$. Denote this elements of a new population $y_1, \ldots, y_{2P}$.

3. Crossover. We randomly choose two vectors $\tilde{x}_j$ and $\hat{x}_k$ in the set $\{x_{m_{2P+1}}, \ldots, x_{m_{P^2}}\}$ and breed them to produce a 'child'. In order to do this we construct the l-th element ($l = 1, \ldots, N$) of the new vectors $a_i = (a_{i1}, \ldots, a_{iN})$, $i = 2P + 1, \ldots, P^2$, $a_i \in D_K$, from vectors $\tilde{x}_j$ and $\hat{x}_k$, $\forall j, k = 2P + 1, \ldots, P^2$, by choosing between $\tilde{x}_{jl}$ and $\hat{x}_{kl}$ following the rules:

- if $\tilde{x}_{jl} = \omega_j > 0$ and $\hat{x}_{kl} = \omega_k > 0$ (i.e. the asset is in both parents portfolios), than the asset in the child is as follows $a_{il} = \chi \cdot \omega_j + (1 - \chi) \cdot \omega_k$, where χ is randomly generated number in [0, 1];
- if $\tilde{x}_{jl} = 0$ and $\hat{x}_{kl} = 0$ (i.e. the asset is not in either parent portfolios), than $a_{il} = 0$ (this asset is not in the child);
- if $\tilde{x}_{jl} = \omega_j > 0$ and $\hat{x}_{kl} = 0$ (i.e. the asset is in only one of the parent portfolios), than with probability π $a_{il} = \omega_j$ (i.e. this asset is included in the portfolio with probability π).

Although there are lots of other crossover operators known in the literature (Haupt and Haupt 2004, Cormen *et al.* 2010, Abdoun and Abouchabaka 2011) our implementation and testing of the genetic algorithm approach with developed crossover stage (specially adjusted for prospect theory portfolio selection problem) have shown good convergence and improvement of the solution in each generation. Neither divergence nor cyclic errors has been detected during the algorithm's progress. A simple approach provides appropriate CPU time, convenient pseudo-code and programme code which is flexible enough to play with parameters and conditions of the models.

4. Mutation. To introduce mutation we change each element of the constructed vector a_i with a given small probability $\zeta > 0$ for the randomly generated number from [0,1]. Then we ensure that the number of non-zero elements of the new vector is less than or equal to K and normalize the elements of this vector. Doing empirical experiments, we have noticed that genetic algorithm choose not too diversified portfolios and very rare the number of assets in the chosen portfolios exceed cardinality constraint much. So, if in some cases the cardinality constraint condition is broken, we randomly choose non-zero elements of the vector one by one in order to make it zeros till $\sum_{i=1}^{N} \varphi_i \leq K$. After checking the K condition we normalize the vector.

We also find the maximum of the vectors $a_i, \tilde{x}_j, \hat{x}_k$ and denote this as y_i. This is the most fit vector and now move this to the new population. Continue while the last y_{P^2} element of the new population matrix have been processed.

5. Assessment. We calculate the values $\mathrm{PT} + \mathrm{IT}_{cc}(y_1), \ldots, \mathrm{PT} + \mathrm{IT}_{cc}(y_{P^2})$ and compare the maximum values of the obtained objective function $\max\mathrm{PT} + \mathrm{IT}_{cc}(y_i)$ to $\max\mathrm{PT} + \mathrm{IT}_{cc}(x_i)$. The new population proceeds to the new generation (if $g < G$) if and only if $\max\mathrm{PT} + \mathrm{IT}_{cc}(y_i) \geq \max\mathrm{PT} + \mathrm{IT}_{cc}(x_i)$.

6. Final Assessment. In the last generation $g = G$ find the vector $y_i^* = \{y_i \mid \max\{\mathrm{PT} + \mathrm{IT}_{cc}(y_1), \ldots, \mathrm{PT} + \mathrm{IT}_{cc}(y_{P^2})\}$. The vector y_i^* then is the best solution.

Pseudo-code of the genetic algorithm for prospect theory utility function maximization is given below.

```
Generate initial population x_i ∈ D_K, i = 1, ..., P^2,
cycle of G generations
    calculate values PT+IT_cc(x_1), ..., PT+IT_cc(x_P2)
    sort PT + IT_cc(x_m1) ≥ ... ≥ PT + IT_cc(x_mP2)
    save maxPT + IT_cc(x_i)
    x_m1, ..., x_m2P = y_1, ..., y_2P
    proceed to the next generation
        randomly pick x̃_j and x̂_k in the set
        {x_m2P, ..., x_mP2}
        ∀i, j, k, l, i, j, k = 2P + 1, ..., P^2, l = 1, ..., N
            if x̃_jl = ω_j and x̂_kl = ω_k
                then a_il = χ · ω_j + (1 − χ) · ω_k,
```

$\chi \in U(0, 1)$
else if $\tilde{x}_{jl} = 0$ and $\hat{x}_{kl} = 0$
then $a_{il} = 0$
else if $\tilde{x}_{jl} = \omega_j$ and $\hat{x}_{kl} = 0$
then with $\pi a_{il} = \omega_j$
with mutation probability $\zeta > 0$
$a_{il} \leftarrow \hat{a}_{ij}$, $\hat{a}_{il} \in U(0, 1)$
choose $\max\{\mathrm{PT} + \mathrm{IT}_{cc} : y_i \in \{a_i, \tilde{x}_j, \hat{x}_k\}\}$
find $\mathrm{PT}+\mathrm{IT}_{cc}(y_i) = \max\{\mathrm{PT}+\mathrm{IT}_{cc}(y_1), \ldots, \mathrm{PT}+\mathrm{IT}_{cc}(y_{P2})\}$
choose $\mathrm{PT} + \mathrm{IT}_{cc}(y_i^*) = \max\{\max\mathrm{PT} + \mathrm{IT}_{cc}(y_i), \max\mathrm{PT} + \mathrm{IT}_{cc}(y_{P2})\}$
y^*i is an optimal solution

5. Computational investigations

5.1. Data

We have solved the portfolio optimization problems using publicly available data relating to five major market indices, available from the OR-Library (Beasley 2003). The five market indices are the Hang Seng (Hong Kong), DAX 100 (Germany), FTSE 100 (UK), S&P 100 (USA) and the Nikkei 225 (Japan) for 290 time periods each (weekly data), taken from: http://people.brunel.ac.uk/mastjjb/jeb/orlib/portinfo.html. All of these problems were considered previously by Chang *et al.* (2000) and Woodside-Oriakhi *et al.* (2011). The size of these five test problems ranged from $N = 31$ (Hang Seng) to $N = 225$ (Nikkei 225) and are presented in table 1.

The data used in this paper is given in the form of matrices of asset prices. We transformed the original data-sets into matrices of asset log returns. It is widely accepted to use logarithm of the price ratio in order to derive the rate of returns, instead of using absolute asset price relations (Morgan 1996). In our research, the rate of return r is calculated using the prices p for each time period s as follows:

$$r_i = \ln\left(\frac{p_{is}}{p_{is-1}}\right), i = 1, \ldots, N, s = 1, \ldots, S,$$

where N is the number of assets and S is the total number of time periods.

In this research, we apply simulation of the data with a particular type of distribution as an out-of-sample test data for our models. We are interested in so called 'bullish' market dynamics which indicates the investor's confidence that the positive trend of the prices will continue. It also characterizes increasing investments and high activity of exchange trades which follows from a stable economic situation. In contrast a 'bearish' market demonstrates pessimistic expectations which leads to stagnation and long-term decreasing of the prices. In order to investigate the performance of the models in different conditions we simulate these two trends in the matrix of the asset returns.

Table 1. Test problem dimension.

	Data-set	Number of stocks N	Number of time periods S	K
1	Hang Seng	31	290	15
2	DAX 100	85	290	20
3	FTSE 100	89	290	25
4	S&P 100	98	290	25
5	Nikkei 225	225	290	25

The out-of-sample data-set which simulates bullish and bearish markets were obtained using the built in functions available in the Statistics Toolbox in Matlab. For bullish market simulations we apply the function *datasample*. This function $y = datasample(data, k)$ returns k observations sampled uniformly at random, with replacement, from the specific data-set in $data$. In order to obtain the data-set which possesses properties of a bullish market we simulate the returns based on historical data of market growth (data from 4.01.2005 to 30.12.2005; 252 time periods in total).

Bearish market simulations are made using the command *mvtrnd*. The statement $r = mvtrnd(kR, df, cases)$ returns a matrix of random numbers chosen from the multivariate t-distribution, where kR is matrix of historical returns from the crisis period, df is the degrees of freedom (in our computational study $df = 5$) and it is either a scalar (like we use in this research) or could be a vector with cases elements (case is the number of lines, equal to 100 for these tests). We chose a t-distribution because the tails of a Student t-distribution tend to zero slower than the tails of the normal distribution which reflects more the real market situation. For the simulations of bearish market we used historical data related to the FTSE 100 index of the global crisis period in 2008 available in Bloomberg Database (data from 1.01.2008 to 31.12.2008; 261 time periods in total) as an initial matrix for simulation. So we apply both, crisis historical data as a sample of data and a t-distribution simulation in order to underline the contrast in two different types of return distributions, bullish and bearish.

The index tracking portfolio selection problem (basic formulation and with cardinality constraints) were solved using AMPL software with CPLEX (version 12.5.1.0) as a software package for solving large-scale optimization problems. The prospect theory with index tracking portfolio selection problem (without and with cardinality constraints) were implemented using Matlab software, as well as built-in and specially developed functions. All simulations were run in Matlab. The system runs under MS Windows 7 64-bit SP 1 and in our computational work we used an Intel Core i3-2310M pc with a 2.10 GHz processor and 8.0 GB RAM.

5.2. Parameters of the models

For the basic prospect theory model (also with index tracking and with cardinality constraints) we use constant values of the parameters $\lambda = 2.25$, $\alpha = \beta = 0.88$ as proposed by Tversky and Kahneman in their paper (Tversky and Kahneman 1992). Tversky and Kahneman consider prospect theory as a complex choice model. Estimation of such types of problems is very difficult because of the large number of parameters. In order to reduce this number they 'focused on the qualitative properties of the data rather than on parameter estimates and measures of fit' (Tversky and Kahneman 1992) using a nonlinear regression procedure for estimation of the parameters of equation (12), they found that 'the median exponent of the value function was 0.88 for both gains and losses, in accordance with diminishing sensitivity' and 'the median λ was 2.25' (Tversky and Kahneman 1992).

We used K as a parameter for cardinality constrained models as shown in table 1. These models according to its formulation have lower and upper limits on the asset weight. We use $l_i = 0.01$ and $u_i = 1$ for these limits.

5.3. *Parameters of the heuristic approaches*

Previously we note that prospect theory model (also with index tracking) is mathematically complex problem and therefore it is difficult to deal with. In section 2, we proposed different solution approaches to this model. In order to obtain an 'optimal' solution for the behaviourally based model (basic formulation and with index tracking) we use a differential evolution algorithm and a genetic algorithm.

It is known that the parameters of heuristics and metaheuristic algorithms have a great influence on the effectiveness and efficiency of these algorithms (Akbaripour and Masehian 2013). It is important to find correct parameter settings for each problem and data-set. To obtain the best solution for the problems we illustrate this here with the algorithms using the first data-set (Hang Seng) trying to choose the most appropriate value for each parameter and analyse the effectiveness of each algorithm in order to define the best for our research. We tested effectiveness of both algorithms applied to basic prospect theory problem (14)–(17). The analysis and selection of the parameters for the chosen algorithm for the other sets of data are presented in Appendix 1.

Our choice of parameter is based on three comparison criteria: computational time, utility as the value of the objective function $\mathrm{PT}(x)$ and range of $\mathrm{PT}(x)$ as a difference $\xi = \max \mathrm{PT}(x) - \min \mathrm{PT}(x)$. In order to study the stability of the algorithm we test each combination of parameters 10 times and compare mean CPU time, mean utility and ξ in the form of the difference $\max \mathrm{PT}(x) - \min \mathrm{PT}(x)$.

The optimal solution of the prospect theory problem is typically unknown and we have no benchmark for comparative analysis. So we define the optimal solution to be the best in the set of solutions we have obtained in our tests.

Much research has been devoted to using heuristic approaches as an effective tool for dealing with non-convex problems. Maringer in 2008 presented a comparative analysis of quadratic, power and the prospect theory utility function performance with different levels of loss aversion (Maringer 2008). He used a differential evolution approach in order to get a solution for the prospect theory model. The paper focused more on performance of the models and parameters of the optimal portfolio return distribution but not on the solution approach itself.

To the best of our knowledge there are no studies where the genetic algorithm has been applied to the prospect theory problem. From the mathematical point of view it is interesting to investigate the performance of different solution approaches applied to problem (14)–(17) also with additional constraints and index tracking modification which is non-convex and function (14) which is non-differentiable.

Differential evolution algorithm

The differential evolution algorithm efficiency depends on parameters such as the differential weight F, the crossover probability CR, the population size P and the number of generations G. It is necessary to start with the F parameter because the differential weight is the key parameter for the differential evolution algorithm. As we noticed this value significantly influences the mean value of the objective function and its dispersion. It is known that $F \in [0, 2]$ (see section 4.1), however, in our case a value larger than 1 gives us a very unstable solution. Thus, we define the following values to test: 0.05, 0.15, 0.5 and 0.95. In the calculations shown in table 2 for our specific function, the smaller the value of the differential weight the higher the value of objective function (utility) and the smaller the range of the solution ($\xi = 0$ leads to the best quality of the solution). The value 0.05 gives us the best results according to all three criteria.

It should be mentioned that in choosing parameter $F = 0.05$ we set $\mathrm{CR} = 0.5$, $P = 20$ and $G = 100$. This choice is based on preliminary analysis and recommendations available in the literature (Price *et al.* 2005, Feoktistov 2006). Hereinafter, while testing each parameter one by one we fix the values of other parameters ($F = 0.05$, $\mathrm{CR} = 0.5$, $P = 20$ and $G = 100$) in order to show the difference in the results. In table 2 and further tables the chosen best parameters indicated in bolt font.

The next step is to choose the optimal value for the crossover probability. It is known that the $\mathrm{CR} \in [0, 1]$ (see section 4.1). We analyse three values for $\mathrm{CR} = 0.3, 0.5, 0.8$. The results in table 2 confirms that $\mathrm{CR} = 0.5$ provides an acceptable CPU time (better than $\mathrm{CR} = 0.8$) and a stable utility (better than $\mathrm{CR} = 0.3$) which leads to a stable solution.

The parameters F and CR should be chosen for the specific objective function and features of the problem. In contrast, the values of G and P primarily depend on the size of the problem. For example, for a data-set with 31 assets we define values for G and P, so, for larger scale problems we use values in proportion to the best we find here. We consider the values of these parameters as a function of problem size. We now explain the choice of these parameters only for the smallest data-set Hang Seng.

We test values $P = 15, 20, 25$ in order to define suitable parameters in terms of CPU time and optimality of the solution. As one can see in table 2 the population size of 20 provides the best utility (quantitatively and in terms of stability) with reasonable computational time. The value $P = 25$ requires more time (+35.6 s) compared to $P = 20$, providing the same utility while a smaller population size leads to an unstable solution.

Within the DE algorithm we need to decide which number of generations is the best for this problem size. We define three points to test which are $G = 70, 100, 130$ in order to find a balance between solution quality and computational time. We choose 100 because it provides maximum utility with range 0 in an acceptable CPU time as shown in table 2.

Genetic algorithm

There are three main parameters in the genetic algorithm: the mutation probability z, the population size P and the number of generations G. These parameters are the most influencing on the outcome of the algorithm.

As shown in table 3 we tested different values for each of these parameters in order to find the optimal settings. In the analysis we used constant parameters $z = 0.5$, $P = 15$ and $G = 70$ for the Hang Seng (Hong Kong) data-set while testing each parameter in order to show the difference in the results. This choice is based on preliminary analysis and recommendations available in the literature.

Table 2. Differential evolution parameter comparison (Hang Seng data set).

Parameter	Parameter value	CPU time	PT(x)	ξ
F	**0.05**	**61.8**	**0.6237**	**0**
	0.15	64.4	0.6235	0.0003
	0.5	66	0.62084	0.0013
	0.95	69.2	0.56534	0.0269
CR	0.3	61.6	0.62356	0.0002
	0.5	**61.8**	**0.6237**	**0**
	0.8	65.4	0.6237	0
P	15	35.2	0.62302	0.0031
	20	**61.8**	**0.6237**	**0**
	25	97.4	0.6237	0
G	70	43.2	0.62342	0.0005
	100	**61.8**	**0.6237**	**0**
	130	80.6	0.6237	0

First of all the mutation probability should be chosen. We took several different values for the parameter z. As one can see in table 3 the CPU time does not change much and does not depend on the value of this parameter. It is obvious that $z = 0.5$ gives us a necessary and sufficient mutation component to obtain the best stability of the solution. The values larger ($z = 0.7$) or smaller ($z = 0.3$) provide the solution with lower level of stability. In addition, the value of the objective function in this case is not the best as well.

Population size is a very important parameter for any heuristic algorithm. One should find the right value of P for the specific problem. There are many recommendations in the literature which can help to choose suitable parameters for the genetic algorithm (Fogel 2006) according to the specific objective function. Most of the guides suggest to use the number of variables and multiply it by 10 for such complex objective functions such as prospect theory utility function. At the same time for the portfolio optimization problem the recommended population size is around 100–200 (Alander 1992). In our case, there are 31 assets in a data-set and we found testing the model that reasonable interval for the search is [10, 20] for such a small matrix. Taking into account that in our algorithm we use population size P^2 we obtained an interval [100, 400] which covers the first recommendation ($31 \times 10 = 310$) and the second one [100, 200].

The population size greatly affects the CPU time. Again we are searching for a balance between computational time and stability because the quality is not improving much with an increasing value of P. However, the solution becomes more volatile once you decrease the population size (see results for $P = 10$ in table 3). We define $P = 15$ as the best for our experiments because it gives optimal utility and saves computational time compared to $P = 20$. Also $P = 15$ provides a good search space for exploration.

We study the interval [40,100] in order to define the optimal parameter value for the number of generations. Previously, we tested extremely high values such as 300 and 400 and the quality of the solution did not change much versus the value of 100 but the CPU time increases dramatically. One can see in table 3 that the difference between the results obtained using $G = 70$ and $G = 100$ is not much too, so, we can save time for approximately the same range of the solution and the value of objective function while decreasing the value of G results in a deterioration solution.

As was mentioned previously, we consider values of P and G parameters as a function of the problem size for the heuristic approaches and one should choose it proportionally to the problem size. The values of G and P parameters for the genetic algorithm for different sized problems can be found in Appendix 1.

It is important to note that both different algorithms give us the same value of the objective function. This fact verifies the solution obtained with the proposed solution approaches and confirms the accuracy of the implementation of the prospect theory model into heuristic approaches.

We notice that the value of criterion ξ for the genetic algorithm is slightly worse than the results achieved when testing the differential evolution algorithm. At the same time the CPU time of the GA is much less which gives a benefit compared to the DE. This benefit defines the choice of this solution approach for further computational study for this research.

5.4. The index tracking problem and prospect theory model

The index tracking problem usually chooses many assets in the optimal portfolio which is very difficult to manage and rebalance. That is why the IT has a cardinality constraint which then becomes a computationally challenging problem for researchers. In this section, we discuss empirical results of in-sample and out-of-sample performance of the IT and PT with index tracking problems (with and without cardinality constraint). As out-of-sample tests we use simulation of bullish and bearish market.

The computational results presented in this section for index tracking problems were obtained using five data-sets described earlier. The first asset in each data-set is the index and is not included in the investment universe of assets. We also use a methodology described above for simulation of bullish and bearish markets in out-of-sample tests.

We analyse the performance of the results by several criteria such as CPU time, the number of assets in the portfolio n, tracking error TE, TE over the index TE o, TE under the index TE u. It should be noted that we use absolute values of TE, TE o, TE u for our analysis. Table 4 reflects the empirical results of the experiment for the used sets of data.

It is easy to see from the table that the number of assets in the PT with IT optimal portfolios is approximately half those in the IT portfolios. This issue gives a good advantage to the PT

Table 3. Genetic algorithm parameter comparison (Hang Seng data set).

Parameter	Parameter value	CPU time	PT(x)	ξ
z	0.3	36.6	0.6219	0.0084
	0.5	**36.2**	**0.62354**	**0.0002**
	0.7	36.8	0.62352	0.0004
P	10	15.6	0.60916	0.0713
	15	**36.2**	**0.62354**	**0.0002**
	20	67.4	0.62361	0.0002
G	40	25.6	0.6235	0.0034
	70	**36.2**	**0.62354**	**0.0002**
	100	47.2	0.62358	0.0001

with IT in comparison with the IT model because of transaction costs and convenience of portfolio management.

It is obvious that the TE of the IT model solution is always less than in PT with IT optimal portfolios but it is still comparable. One can notice that the beneficial difference between parameters TE o for IT and PT with IT models is much greater (in proportion to the TE) than between parameters TE u for these models. This means that the PT with IT model chooses assets with higher return than the IT model using the reference point (index) only as a starting point but not as a benchmark. These facts confirm that the PT with IT model focuses more penalty on not achieving the reference point compared with exceeding it.

We test the performance of the two models using out-of-sample simulations and use the same criteria for analysis. Firstly, we simulate on a bullish market. Table 5 reflects the empirical results of the experiment.

We should note that the behaviour of the investigated models in the bullish market is very similar to the in-sample performance. According to the TE parameter the PT with IT portfolios show smaller value compare to the in-sample results.

We also test the performance of two models using an out-of-sample simulation on a bearish market. It is interesting to explore the performance of the models in opposite conditions. In table 6 one can find the out-of-sample empirical results.

In contrast with the previous results, PT with IT model fails to show a good outcome. This model performs worse in each data-set for each parameter when compared to the IT. Only TE of the prospect theory with IT improved and becomes even less than for IT model portfolios.

Finally, we can conclude that the prospect theory model with index being a reference point is very effective in an increasing market due to its mathematical formulation which makes it desirable to exceed the reference point (in our case it is the index values). In addition it is more beneficial in terms of lower number of assets in the optimal portfolio. However, in a crisis market situation PT with IT model performs worse than IT. Thus, the prospect theory model adjusted for index tracking works well in a stable or increasing market condition.

Cardinality constrained index tracking and prospect theory with index tracking models

The index tracking model with a cardinality constraint is a very computationally challenging problem. On the one hand, the optimal solution is unknown and one should set the termination criteria very carefully to obtain the best results. On the other hand, the CPU time required is significantly large versus the non-cardinality constrained model.

For the index tracking and prospect theory with index tracking models with cardinality constraint we used similar asset thresholds $l_i = 0.01$, $u_i = 1$ $(i = 1, \ldots, N)$ and parameter K which is the number of assets allowed to be included in the optimal portfolio as described in section 5.2.

Tables 7–9 show the performance of the IT and PT with IT models with the cardinality constraint in-sample, out-of-sample (simulation of bullish market) and out-of-sample (simulation of bearish market) empirical results.

As displayed in the tables the behaviour of the models with the cardinality constraint is completely similar to the behaviour of the non-cardinality constrained IT and PT with IT models in different conditions. It should be noted that CPU time for behavioural models with the additional constraint does not change much and it implies that the genetic algorithm deals well with such type of complex problems. So, the cardinality constrained models results confirms the conclusion about the character of compared models made above.

5.5. Summary

In this section, the empirical study and analysis are presented. We discuss the parameters of the models as well as define parameters for developed heuristic algorithms applied to the prospect theory model. We mentioned above that using heuristic solution approaches the parameters of these algorithms is very important for an accurate solution.

Previously, prospect theory model was considered by other researchers in the literature. In contrast to similar researches (Levy and Levy 2004 and Pirvu and Schulze 2012) known in the literature we obtained the optimal portfolios for the prospect theory with index tracking model independently and not as a subset of the compared (index tracking) model efficient set. In addition, we use data with different types of asset returns distribution but not normal in contrast to Levy and Levy (2004).

In unpredictable market conditions the index tracking portfolio selection problem becomes very popular. We investigated the prospect theory model with the index as the reference point (with and without cardinality) compared to the basic index tracking model. It has been found that PT model is more beneficial in terms of lower number of assets in the portfolio than index tracking (for models without cardinality constraint) that reduces transaction costs and makes rebalancing of the

Table 4. Comparative analysis of the index tracking and prospect theory with index tracking problem (in-sample).

Data-set	Model	CPU time	n	TE	TE o	TE u
Hang Seng	IT	0.047	30	0.4290	0.2444	0.1845
	PT+IT	70	20	0.8420	0.5690	0.2730
DAX 100	IT	0.109	69	0.3354	0.1835	0.1519
	PT+IT	242	51	1.1763	0.7336	0.4427
FTSE 100	IT	0.141	81	0.2855	0.1657	0.1198
	PT+IT	250	46	1.1463	0.7919	0.3544
S&P 100	IT	0.125	83	0.2682	0.1553	0.1130
	PT+IT	347	67	0.9409	0.5881	0.3529
Nikkei 225	IT	0.266	159	0.1686	0.0921	0.0765
	PT+IT	1803	69	0.9802	0.6300	0.3501

Table 5. Comparative analysis of the index tracking and prospect theory with index tracking problem (out-of-sample: simulation of bullish market).

Data-set	Model	TE	TE o	TE u
Hang Seng	IT	0.1292	0.1292	0
	PT+IT	0.3589	0.3589	0
DAX 100	IT	0.0934	0.0918	0.0016
	PT+IT	0.5470	0.5470	0
FTSE 100	IT	0.1304	0.1304	0
	PT+IT	0.6335	0.6335	0
S&P 100	IT	0.1271	0.1271	0
	PT+IT	0.4432	0.4432	0
Nikkei 225	IT	0.1225	0.1225	0
	PT+IT	0.5660	0.5660	0

Table 6. Comparative analysis of the index tracking and prospect theory with index tracking problem (out-of-sample: simulation of bearish market).

Data-set	Model	TE	TE o	TE u
Hang Seng	IT	0.1960	0.1960	0
	PT+IT	0.1806	0.1806	0
DAX 100	IT	0.2991	0.1673	0.1317
	PT+IT	0.2928	0.1481	0.1446
FTSE 100	IT	0.3013	0.1217	0.1795
	PT+IT	0.3136	0.1164	0.1972
S&P 100	IT	0.3026	0.1172	0.1854
	PT+IT	0.2984	0.1085	0.1899
Nikkei 225	IT	0.2750	0.1342	0.1408
	PT+IT	0.3017	0.0880	0.2137

Table 7. Comparative analysis of index tracking and prospect theory with index tracking problem with cardinality constraint (in-sample).

Data-set	Model	CPU time	K	n	TE	TE o	TE u
Hang Seng	IT_{cc}	102	15	15	0.5760	0.3316	0.2448
	$PT + IT_{cc}$	74	15	15	1.1871	0.7828	0.4044
DAX 100	IT_{cc}	200	20	20	0.5889	0.3280	0.2609
	$PT + IT_{cc}$	275	20	20	1.3309	0.9616	0.3693
FTSE 100	IT_{cc}	193	25	25	0.6650	0.3819	0.2831
	$PT + IT_{cc}$	323	25	24	1.4432	1.0323	0.4109
S&P 100	IT_{cc}	176	25	25	0.5555	0.3223	0.2332
	$PT + IT_{cc}$	459	25	22	1.2972	0.9111	0.3861
Nikkei 225	IT_{cc}	612	25	25	0.7211	0.3845	0.3367
	$PT + IT_{cc}$	2780	25	25	1.3179	0.9637	0.3542

portfolio more convenient. We also noticed that returns of the PT with index tracking model mostly exceed the index returns which confirms our previous conclusion about the impact of the reference point. However, in a bearish market the prospect theory model shows greater losses compared to the index tracking model.

Table 8. Comparative analysis of index tracking and prospect theory with index tracking problem with cardinality constraint (out-of-sample: simulation of bullish market).

Data-set	Model	TE	TE o	TE u
Hang Seng	IT_{cc}	0.1519	0.1519	0
	$PT + IT_{cc}$	0.3915	0.3915	0
DAX 100	IT_{cc}	0.1202	0.1195	0.0007
	$PT + IT_{cc}$	0.7190	0.7190	0
FTSE 100	IT_{cc}	0.1826	0.1826	0
	$PT + IT_{cc}$	0.7285	0.7285	0
S&P 100	IT_{cc}	0.1674	0.1674	0
	$PT + IT_{cc}$	0.6149	0.6149	0
Nikkei 225	IT_{cc}	0.1296	0.1296	0
	$PT + IT_{cc}$	0.6326	0.6326	0

Table 9. Comparative analysis of the index tracking and prospect theory with index tracking problem with cardinality constraint (out-of-sample: simulation of bearish market).

Data-set	Model	TE	TE o	TE u
Hang Seng	IT_{cc}	0.2484	0.2484	0
	$PT + IT_{cc}$	0.1992	0.1992	0
DAX 100	IT_{cc}	0.2907	0.1761	0.1145
	$PT + IT_{cc}$	0.3343	0.1652	0.1691
FTSE 100	IT_{cc}	0.3248	0.1165	0.2083
	$PT + IT_{cc}$	0.3268	0.0954	0.2313
S&P 100	IT_{cc}	0.2795	0.1208	0.1586
	$PT + IT_{cc}$	0.3066	0.0876	0.2189
Nikkei 225	IT_{cc}	0.2939	0.1720	0.1219
	$PT + IT_{cc}$	0.3366	0.0943	0.2423

6. Conclusion

The behavioural approach to portfolio theory has become very popular in the last decade because the market has demonstrated significant instability. There is much theoretical evidence in the literature that behaviourally based models could help to decrease the risk of the portfolio since they take into account natural loss aversion and risk aversion biases of the investors. However, we found that there is a lack of practical and empirical studies in the literature which could show and prove these benefits and shed light on the performance of these models in different market situations.

In this research, we studied a behaviourally based model namely the prospect theory with index tracking model, using a comparative analysis with the traditional index tracking model. In order to investigate the benefits of a behavioural approach we implemented cardinality constraints in these models and tested the results out-of-sample using simulations of bullish and bearish return distributions. The results were presented for five publicly available data-sets which reflect the dynamics of major world markets.

We developed several solution approaches for the prospect theory model to obtain an accurate solution using heuristics. The differential evolution algorithm and the genetic algorithm were implemented in Matlab in order to do this. We also justify the parameter choice for these models using an empirical study due to the importance of the parameters in heuristic algorithm applications. However, we propose some limitations on suggested solution approaches. These are mainly connected with the increasing size of problem data-sets and they affect CPU time and convergence of the algorithms. In order to use these algorithms for bigger data-sets, the generalization of some algorithm stages should be done (for example, regarding crossover and mutation).

The application of the prospect theory with index tracking model to the portfolio optimization problem shows that the model obtains higher returns in comparison with the basic index tracking model. This can be explained by the effect of the reference point. Prospect theory wants to exceed the reference point (for example, the risk-free rate) as much as possible reflecting psychological biases. So, this reference point steers the model to choose the assets with higher returns no matter at which desired level of return for the whole period is set.

We can conclude that prospect theory optimal portfolios performed better in terms of returns than the index tracking model and the index itself, both in-sample and in a bullish market. However, the PT model was slightly worse in a bearish market compared to the index tracking model. At the same time it has been found that the PT with IT model is normally less diversified than the IT model which is a benefit in terms of transaction costs and portfolio management issues.

We would like to point out that in this paper prospect theory was applied to a large universe of assets. Previously, only small experiments were presented in the literature (for example 2–3 assets). Thus, this empirical study aims to encourage the use of prospect theory in practice along with mean variance and index tracking models for specific real market conditions.

At the same time, the use of a behaviourally based approach to the portfolio selection problem has potential limitations in applications to derivatives because it is difficult to implement different conditions and types of the contracts. The question here is how to implement additional information in to the model and how to identify the influence of the behavioural biases to the results in the analysis. In addition, it is not clear which solution approach can be applied to this problem.

It should be noted that the problem of portfolio optimization using a behavioural approach is very challenging. There are different ways to investigate its solution and performance.

As was proposed in this research, we developed several heuristic solution approaches for the prospect theory model taking into account the specific features of the model. As an idea for future work, one can bring more intelligent choice of the assets in the portfolio into the breeding stage of the genetic algorithm based on the observations and preferences of the studied model. In each generation one distinguishes the assets which are included in the best portfolio and use this information for the breeding stage in the next generation. Instead of checking all assets in the data, the algorithm could find the preferable one faster than using the information about frequency of appearance of assets in previous best portfolios. It could help to decrease the CPU time for this algorithm by reducing the search space of suitable assets for the best portfolios and decreasing the number of generations.

Funding

The first author was part funded by RFBR [grant number 14-01-00140]. Dr Nina Grishina express gratitude to Saratov State University, especially to Prof. Leonid Kossovich, Prof. Tatiana Zakharova and Prof. Sergei Sidorov, for their significant help in this work.

References

Aarts, E.H.L. and Lenstra, J.K., *Local Search in Combinatorial Optimization*, 2003 (Princeton University Press: Princeton, USA).
Abdoun, O. and Abouchabaka, J., A comparative study of adaptive crossover operators for genetic algorithms to resolve the traveling salesman problem. *Int. J. Comput. Appl.*, 2011, **31**, 49–57.
Akbaripour, H. and Masehian, E., Efficient and robust parameter tuning for heuristic algorithms. *Int. J. Ind. Eng. Prod. Res.*, 2013, **24**, 143–150.
Alander, J.T., On optimal population size of genetic algorithms. In *Proceedings of the Computer Systems and Software Engineering Comp Euro*, pp. 65–70, 1992 (IEEE Computer Society Press: The Hague).
Allais, M., Le comportement de lhomme rationnel devant le risque: Critique despostulats et axiomes de lecole Americaine. *Econometrica*, 1953, **21**, 503–546.
Ayodele, A.A. and Charles, K.A., Portfolio selection problem using generalized differential evolution 3. *Appl. Math. Sci.*, 2015, **9**, 2069–2082.
Barberis, N. and Huang, M., Mental accounting, loss aversion, and individual stock returns. *J. Finance*, 2001, **56**, 1247–1292.
Beasley, J.E., *Population Heuristics*, *Handbook of Applied Optimization,* 2002 (Oxford University Press: Oxford).
Beasley, J.E., OR-Library. Brunel.ac.uk, 2003. Available online at: http://people.brunel.ac.uk/mastjjb/jeb/orlib/indtrackinfo.html (accessed 27 January 2015).
Beasley, J.E., Meade, N. and Chang, T.J., An evolutionary heuristic for the index tracking problem. *Eur. J. Oper. Res.*, 2003, **148**(3), 621–643.
Bernoulli, D., Exposition of a new theory on the measurement of risk. *Econometrica*, 1954, **22**, 23–36.
Camerer, C. and Ho, T.H., Violations of the betweenness axiom and nonlinearity in probability. *J. Risk Uncertainty*, 1994, **8**(2), 167–196.
Chang, T.J., Meade, N., Beasley, J.E. and Sharaiha, Y.M., Heuristics for cardinality constrained portfolio optimisation. *Comput. Oper. Res.*, 2000, **27**, 1271–1302.
Chang, T.J., Yang, S.C. and Chang, K.J., Portfolio optimization problems in different risk measures using genetic algorithm. *Expert Syst. Appl. Int. J.*, 2009, **36**(7), 10529–10537.
Cormen, T.H., Leiserson, C.E., Rivest, R.L., and Stein, C., *Introduction to Algorithms*, 3rd ed., 2010 (MIT Press: Cambridge).
Curtis, G., Modern portfolio theory and behavioral finance. *J. Wealth Manage.*, 2004, **7**, 16–22.
Das, S., Markowitz, H., Scheid, J. and Statman, M., Portfolio optimization with mental accounts. *J. Financial Quant. Anal.*, 2010, **45**, 311–334.
di Tollo, G., Hybrid metaheuristic for portfolio selection: Comparison with an exact solver and search space analysis. In *Proceedings of the Proceedings of the Federated Conference on Computer Science and Information Systems*, Vol. 5, pp. 579–588, 2015 (ACSIS: The Lodz).
Dueck, G. and Winker, P., New concepts and algorithms for portfolio choice. *Appl. Stochastic Models Data Anal.*, 1992, **8**, 159–178.
Easley, D. and Yang, L., Loss aversion, survival and asset prices. *J. Econ. Theory*, 2015, **160**, 494–516.
Ellsberg, D., Risk, ambiguity, and the Savage axioms. *Quart. J. Econ.*, 1961, **75**, 643–669.
Evans, J.L. and Archer, S.H., Diversification and the reduction of dispersion: An empirical analysis. *J. Finance*, 1968, **23**, 761–767.
Fama, E., The behaviour of stock market prices. *J. Bus.*, 1968, **38**, 34–105.
Fennema, H. and Wakker, P., Original and cumulative prospect theory: A discussion of empirical differences. *J. Behav. Decis. Making*, 1997, **10**(1), 53–64.
Feoktistov, V., *Differential Evolution: In Search of Solutions*, Optimization and Its Applications, 2006 (Springer: New York).
Fisher, J.H. and Lorie, J.H., Some studies of the variability of returns on investments in common stocks. *J. Bus.*, 1970, **43**, 99–134.
Fogel, D.B., *Evolutionary Computation: Toward a New Philosophy of Machine Intelligence*, 3rd ed., 2006 (Wiley & Sons: Hoboken, NJ).
Gaspero, L.D., di Tollo, G., Roli, A. and Schaerf, A., Hybrid metaheuristics for constrained portfolio selection problems. *Quant. Finance*, 2011, **11**, 1473–1487.
Gomes, F.J., Portfolio choice and trading volume with loss-averse investors. *J. Bus.*, 2005, **78**, 675–706.
Haupt, R.L. and Haupt, S.E., *Practical Genetic Algorithms*, 2nd ed., 2004 (John Wiley & Sons: New York).
Hens, T. and Bachmann, K., *Behavioural Finance for Private Banking*, 2008 (John Wiley & Sons: New York).
Holland, J.H., *Adaptation in Natural and Artificial Systems: An Introductory Analysis with Applications to Biology, Control, and Artificial Intelligence*, 1975 (University of Michigan Press: Ann Arbor, MI).
Homchenko, A.A., Grishina, N.P., Lucas, C. and Sidorov, S.P., Differential evolution algorithm for solving the portfolio optimization problem. *Izvestia Saratov Univ.*, 2013, **13**(2), 88–92.
Jacob, N., A limited-diversification portfolio selection model for the small investor. *J. Finance*, 1974, **29**, 847–856.
Kahneman, D. and Tversky, A., Prospect theory: An analysis of decision under risk. *Econometrica*, 1979, **47**, 263–291.
Karni, E., *Axiomatic Foundations of Expected Utility and Subjective Probability*, Handbook of the Economics of Risk and Uncertainty, 2014 (North Holland: Oxford).
Levy, H. and Levy, M., Prospect theory and mean-variance analysis. *Rev. Financial Stud.*, 2004, **17**, 1015–1041.
Levy, M., Loss aversion and the price of risk. *Quant. Finance*, 2010, **10**, 1009–1022.
Lia, Y. and Yang, L., Prospect theory, the disposition effect and asset prices. *J. Financial Econ.* 2013, **107**, 715–739.
Lin, C. and Liu, Y., Genetic algorithms for portfolio selection problems with minimum transaction lots. *Eur. J. Oper. Res.* 2008, **185**, 393–404.
Lin, D. and Wang, S., A genetic algorithm for portfolio selection problems. *Adv. Model. Optim.*, 2002, **4**, 13–27.

Lopes, L.L., Between hope and fear: The psychology of risk. *Adv. Exp. Soc. Psych.,* 1987, **20**, 255–295.

Mandelbrot, B., The variation of certain speculative prices. *J. Business*, 1963, **36**, 394–419.

Markowitz, H., Portfolio selection. *J. Finance*, 1952, **7**, 7–91.

Markowitz, H., *Portfolio Selection: Efficient Diversification of Investments*, 1959 (John Wiley & Sons: New York).

Maringer, D., Risk preferences and loss aversion in portfolio optimization. In *Computational Methods in Financial Engineering*, edited by E. Kontoghiorghes, B. Rustem, and P. Winker, pp. 27–45, 2008 (Springer: Berlin Heidelberg).

Mitchell, M., *An Introduction to Genetic Algorithms*, 1996 (MIT Press: Cambridge, MA).

Morgan, J.P., *Risk Metrics – Technical Document*, 1996 (J.P. Morgan/Reuters: New York).

Neumann, J. and Morgenstern, O., *Theory of Games and Economic Behavior*, 1944 (Princeton University Press: Princeton, NJ).

Patel, N.G. and Subrahmanyam, M.G., A simple algorithm for optimal portfolio selection with fixed transaction costs. *Manage. Sci.*, 1982, **28**, 303–314.

Peng, H., Gan, M. and Chen, X., A mean-variance model for optimal portfolio selection with transaction costs. In *Proceedings of the 17th World Congress the International Federation of Automatic Control*, Seoul, Korea, 2008 July, 1627–1632.

Pirvu, T.A. and Schulze, K., Multi-stock portfolio optimization under preospect theory. *Math. Financial Econ.*, 2012, **6**, 337–362.

Price, K.V., Storn, R.M. and Lampinen, J.A., *Differential Evolution: A Practical Approach to Global Optimization*, Natural Computing Series, 2005 (Springer: New York).

Reilly, F.K. and Brown, K.C., *Investment Analysis and Portfolio Management*, Vol. 8, 2005 (Thomson South-Western: Mason, OH).

Rieger, M.O. and Wang, M., Prospect theory for continuous distributions. *J. Risk Uncertainty*, 2008, **36**, 83–102.

Roll, R., A mean/variance analysis of tracking error. *J. Portfolio Manage.*, 1993, **18**(4), 13–22.

Rubinstein, M., Markowitz's portfolio selection: A fifty-year retrospective. *J. Finance*, 2002, **57**, 1041–1045.

Schneider, S. and Lopes, L., Reflection in preferences under risk: Who and when may suggest why. *J. Exp. Psychology: Human Percept. Perform.*, 1986, **12**(4), 535–548.

Sengupta, J.K. and Sfeir, R.E., Tests of efficiency of limited diversification portfolios. *Appl. Econ.*, 1985, **17**, 933–945.

Shefrin, H., Prospect theory for continuous distributions. *J. Appl. Corporate Finance*, 2001, **14**, 113–124.

Shefrin, H. and Statman, M., Behavioral portfolio theory. *J. Financial Quant. Anal.*, 2000, **35**, 127–151.

Storn, R. and Price, K., Differential evolution - A simple and efficient adaptive scheme or global optimization over continuous spaces. *J. Global Optim.*, 1997, **11**, 341–359.

Szego, G.P., *Portfolio Theory: With Application to Bank Asset Management*, 1980 (Academic Press: New York).

Tversky, A. and Kahneman, D., Advances in prospect theory: Cumulative representation of uncertainty. *J. Risk Uncertainty*, 1992, **5**, 297–323.

Vlcek, M., Portfolio choice with loss aversion, asymmetric risk-taking behavior and segregation of riskless opportunities. Research paper series No 6–27, Swiss Finance Institute, 2006.

Wang, C.P. and Huang, H.H., Optimal insurance contract and coverage levels under loss aversion utility preference. *Quant. Finance*, 2012, **12**, 1615–1628.

Woodside-Oriakhi, M., Lucas, C. and Beasley, J.E., Heuristics algorits for the cardinality constrained efficient frontier. *Eur. J. Oper. Res.*, 2011, **213**(3), 538–550.

Yang, A.S. and Wu, M.L., Exploring the relationship between investor sentiment and price volatility. *Quant. Finance*, 2011, **11**, 955–965.

Zakamouline, V. and Koekebakker, S., A generalisation of the mean-variance analysis. *Eur. Financial Manage.*, 2009, **15**, 934–970.

Appendix 1. Parameters *G* and *P* of the heuristic approaches

Remark. The parameters of *G* and *P* for the Nikkei 225 data-set is equal to the S&P 100 data-set in our empirical study because specifically for these returns (the Nikkei 225 set) the genetic algorithm finds the best solution quickly enough. So, we do not need to increase the number of generation and population size. The resulting portfolio is undiversified compare to the number of assets available in total. The algorithm defines the preferable assets very fast and the rest of time just plays with the weights.

Table A1. Genetic algorithm parameter comparison for the DAX 100 data set.

Parameter	Parameter value	CPU time	PT(x)	ξ
G	150	444	0.5937	0.0152
	180	**550**	**0.6442**	**0.0002**
	210	652	0.6443	0.0001
P	35	415	0.5694	0.0032
	40	**550**	**0.6442**	**0.0002**
	45	697	0.6437	0.0001

Table A2. Genetic algorithm parameter comparison for the FTSE 100 data set.

Parameter	Parameter value	CPU time	PT(x)	ξ
G	160	532	0.823	0.0043
	185	**630**	**0.8429**	**0.0004**
	220	718	0.8429	0.0002
P	37	479	0.8423	0.0164
	42	**630**	**0.8429**	**0.0004**
	47	755	0.8431	0.0002

Table A3. Genetic algorithm parameter comparison for the S&P 100 data set.

Parameter	Parameter value	CPU time	PT(x)	ξ
G	160	586	0.7353	0.0172
	190	**721**	**0.7822**	**0.0006**
	220	953	0.7853	0.0004
P	40	542	0.7421	0.0043
	45	**721**	**0.7822**	**0.0006**
	50	994	0.7864	0.0003

Table A4. Genetic algorithm parameter comparison for the Nikkei 225 data set.

Parameter	Parameter value	CPU time	PT(x)	ξ
G	160	1050	−0.9555	0.0001
	190	**1179**	−0.9894	**0**
	220	1547	−0.9894	0
P	40	939	−0.9468	0.0021
	45	**1179**	−0.9894	**0**
	50	1486	−0.9894	0